evident all the way to the asymptotic form for large times and distances, for which we have

$$\sigma^\circ = \frac{\sqrt{n}}{2x^\circ}$$

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A RECIPROCAL THEOREM FOR DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY

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A reciprocal principle for dynamic problems of the theory of elasticity is given in the papers [1 to 3]. In this paper, a more general reciprocal principle is presented for the case in which displacement as well as traction boundary conditions are imposed on an elastic body. In contrast to the generally used method of derivation of the reciprocal theorem in dynamics, employing the Laplace transform and Betti's law, the theorem of reciprocity is derived here from a variational principle.

We note that in [4] the opposite route is used for static problems, i. e. the variational principles of the theory of elasticity are deduced from the reciprocal theorem.

Let u_k, v_k, π_k be the components of the displacement, velocity, and generalized momentum vectors: ε_{ik} and σ^{ik} the components of the strain and stress tensors: E^{iklm} the components of the tensor of elastic constants: X^k and P^k the components of the body force and external traction vectors; U^k the components of the specified displacements; and ρ, V and S the density, volume and surface of the elastic body.

The solution of a dynamic problem of the theory of elasticity reduces to the integration of the equations of motion

$$\nabla_i\sigma^{ik} + X^k = \frac{\partial \pi^k}{\partial t} \qquad (1)$$

where

$$\sigma^{ik} = E^{ikjl}\varepsilon_{jl}, \qquad \pi_k = \rho v_k, \qquad \varepsilon_{ik} = \frac{1}{2}(\nabla_i u_k + \nabla_k u_i), \qquad v_k = \frac{\partial u_k}{\partial t} \qquad (2)$$

under the following conditions:

1) the traction boundary conditions

$$\sigma^{ik} n_k = P^i \tag{3}$$

on the part S_1 of the surface S;

2) the displacement boundary conditions

$$u_k = U_k \tag{4}$$

on S_2, the remaining part of S;

3) the initial conditions

$$u_k = u_k^\circ, \qquad \pi_k = \pi_k^\circ \tag{5}$$

at the instant of time $t = 0$.

The problem (1) to (5) in the time interval (0, T) can be represented in the form of an equivalent variational principle

$$\delta J = 0 \tag{6}$$

where

$$J = \int_V \Big\{ -\frac{1}{2} \rho v^i * v_i - \frac{1}{2} E^{ikjl} \varepsilon_{ik} * \varepsilon_{jl} + X^i * u_i - \pi^i (x^k, \tau) [u_i (x^k, 0) - u_i^\circ] +$$

$$+ \pi_{0i} u_i (x^k, \tau) \Big\} dV + \int_{S_1} P^i * u_i \, dS + \int_{S_2} (u_k - U_k) * \sigma^{ik} n_i \, dS \tag{7}$$

and where the following notation has been used:

$$F * G = \int_0^\tau F (x^k, t) \, G (x^k, \tau - t) \, dt \tag{8}$$

To derive the reciprocal theorem, we shall use the fact that if the equations and conditions (1) to (5) are satisfied, then for any choice of variations δu_k the condition (6) is satisfied.

The first variation of the functional (7) can be written in the form

$$\delta J = \int_V \{ -\rho v^i * \delta v_i - E^{ikjl} \varepsilon_{ik} * \delta \varepsilon_{jl} + X^i * \delta u_i -$$

$$- [u_i (x^k, 0) - u_i^\circ] \, \delta \pi^i (x^k, \tau) - \pi^i (x^k, \tau) \, \delta u_i (x^k, 0) + \pi_0^i \delta u_i (x^k, \tau) \} \, dV +$$

$$+ \int_{S_1} P^i * \delta u_i \, dS + \int_{S_2} [(u_k - U_k) * \delta \sigma^{ik} + \sigma^{ik} * \delta u_k] \, n_i \, dS \tag{9}$$

Denoting

$$\delta u_i = u_i' \tag{10}$$

and using Green's theorem, we transform Equation (9) into

$$\delta J = \int_V \Big\{ \Big(\nabla_i \sigma'^{ik} - \frac{\partial \pi'^k}{\partial t} \Big) * u_k + X^i * u_i' + u_i^\circ \pi'^i (x^k, \tau) - \pi'^i (x^k, 0) \, u_i (x^k, \tau) -$$

$$- \pi^i (x^k, \tau) \, u_i' (x^k, 0) + \pi_0^i u_i' (x^k, \tau) \Big\} \, dV +$$

$$+ \int_{S_1} (P^i * u_i' - \sigma'^{ik} * u_k n_i) \, dS - \int_{S_2} (U_k * \sigma'^{ik} - \sigma^{ik} * u_k') \, n_i \, dS \tag{11}$$

Here the functions σ'^{ik}, π'^k are expressed in terms of the functions u'_k (by means of the relations (2)).

If it is now assumed that the functions u'_k are a solution of the elasticity problem (1) to (5) for the body forces X'^k, the external traction P'^k, and the specified displacements U'_k, the following reciprocal theorem is then a consequence of the condition (6) :

$$\int_V [X^i * u_i' + u_i^\circ \pi'^i (x^k,\tau) + \pi_0{}^i u'_i(x^k, \tau)]\, dV + \int_{S_1} p^i * u_i' \, dS -$$
$$- \int_{S_2} U_k * \sigma'^{ik} n_i \, dS = \int_V [X'^i * u_i + u_i'^0 \pi^i (x^k,\tau) + \pi_0'^i u_i(x^k,\tau)]\, dV +$$
$$+ \int_{S_1} P'^i * u_i dS - \int_{S_2} U_k' * \sigma^{ik} n_i dS \qquad (12)$$

Unlike the well-known reciprocal theorem of the dynamic theory of elasticity, the reciprocal relation (12) is also applicable in the case of specified displacements or for a joint specification of traction and displacement boundary conditions.

For statics, the integral (8) assumes the form

$$F * G = \tau F (x^k) \, G (x^k) \qquad (13)$$

and the following reciprocal theorem is obtained from Equation (12)

$$\int_V X^i u_i'^0 \, dV + \int_{S_1} P^i u_i' \, dS - \int_{S_2} U_k \sigma'^{ik} n_i dS = \int_V X'^i u_i dV + \int_{S_1} P'^i u_i dS - \int_{S_2} U'_k \sigma^{ik} n_i dS \qquad (14)$$

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